Log-likelihood ratio (LLR) estimations for Turbo decoders with Quantize-Map-Forward (QMF) Relaying

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Chapter 1

Introduction

1.1 System Model

In the model that is illustrated in Figure 1.1, there is one source and one destination trying to communicate using two relay nodes. We assume that there is no direct link from source to destination. Relay 1 and Relay 2 are half-duplex nodes, which means they cannot transmit and listen at the same time. In order to achieve better diversity and overcome half-duplex limitation on relay nodes, we assume a proper scheduling that will avoid Relay 1 and Relay 2 transmitting at the same time. Using this assumption, we should interpret Figure 1.1 in the following way: Dashed lines in the model are executed at a different time-slot than the straight lines. Basically, we can think as having two independent point-to-point links between source and destination in two time-slots period.

In our model blue circles are representing the Additive White Gaussian Noise (AWGN) channel effects on the transmitted signal. Here we assume that the noise variances at each channel are from the same normal distribution ($\mathcal{N}(0, \sigma^2)$), but they are independent. (RM) block in the figure represents the rate-matching processor in LTE/GSM/etc. Basically, it doesn't have any significance in the following computations, since we assume a deterministic process on each bit. However, for the sake of completeness of the model, we present it in the figure.

We assume that both relays are Quantize-Map-Forward (QMF) relays that *quantize the received signal to the closest point of the source signal constellation map*. This means that relays & destination have the a-priori knowledge of the modulation scheme of the source.

In Equation 1.5, we will define the channel model for one path (Relay i) from source to destination. Similarly, these equations can be written in the same way for the other path (e.g. Relay 1 or 2).

We assume channels with additive white Gaussian noise (AWGN). The noise variances at each channel (η_{sr_i}, η_{rd_i}) are independently chosen from the same normal distribution ($\mathcal{N}(0, \sigma^2)$). The channel equations for one path (Relay RNi)

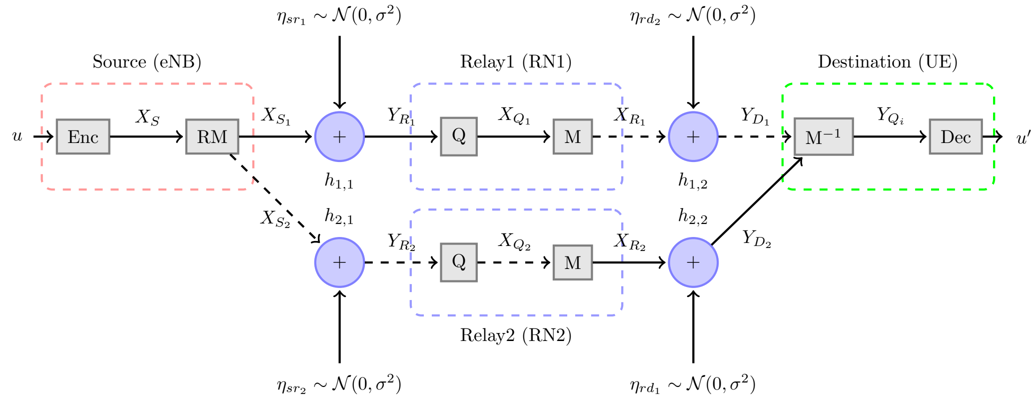

Figure 1.1: Modules used to implement QMF in the diamond network configuration.

from source to destination in Fig. 1.1 is given in (1.1) - (1.5):

$$
\begin{aligned}
Y_{R_i} &= h_{i,1} \cdot X_{S_i} + \eta_{sr_i} && (1.1)\\
X_{Q_i} &= \mathcal{Q}(Y_{R_i}) && (1.2)\\
X_{R_i} &= M(X_{Q_i}) && (1.3)\\
Y_{D_i} &= h_{i,2} \cdot X_{R_i} + \eta_{rd_i} && (1.4)\\
Y_{Q_i} &= M^{-1}(Y_{D_i}) && (1.5)
\end{aligned}
$$

where $\mathcal{Q}$ denotes the quantization operation, M a deterministic mapping function and M^{-1} its inverse.

1.2 Goal

The aim of this document is to compute/estimate the log-likelihood-ratio (LLR) of transmitted source bit ($LLR_{X_{S_i}}$, where $i \in \{1, 2\}$) using the observations Y_{Q_i}at the destination decoder for QMF relaying. Existing conventional practical LLR computations for LTE Turbo decoders do not take into account the existence of QMF relays.

For practical reasons, we are going to look for the simplest representation of $LLR_{X_{S_i}}$as a function of the received observation (Y_{Q_i}).

In the rest of the document, we assume binary signaling (e.g. BPSK), although these computations are also valid for a QPSK signaling. Calculations of QAM constellations can be also done in a similar fashion.

Chapter 2

Log-likelihood Ratio Calculations

2.1 Proposition

Consider a transmitted bit X_{S_i}, and the associated LLR:

$$LLR_{X_{S_i}} = \ln\left(\frac{P(Y_{Q_i} \mid X_{S_i} = +1)}{P(Y_{Q_i} \mid X_{S_i} = -1)}\right) \tag{2.1}$$

Proposition 1 *(2.1) can be approximated as:*

$$LLR_{X_{S_i}} = sign(Y_{D_i} h_{i;2})\ min(\frac{|2Y_{D_i} h_{i;2}|}{\sigma^2}, \frac{|h_{i,1}|^2}{4\sigma^2}) \tag{2.2}$$

where the sign() function returns +1 or -1 depending on the polarity of the value of its parameter, and the notation is as in Fig. 1.1, equations (1.1) - (1.5).

2.2 Proof

Expanding the numerator gives:

$$\begin{aligned}
P(Y_{Q_i} \mid X_{S_i} = 1) &= \sum_{X_{R_i}, Y_{R_i}} P(Y_{Q_i}, X_{R_i}, Y_{R_i} \mid X_{S_i} = 1) && (2.3)\\
&= \sum_{X_{R_i}, Y_{R_i}} [P(Y_{R_i} \mid X_{S_i} = 1) P(X_{R_i} \mid Y_{R_i}, X_{S_i} = 1) \\
&\qquad \cdot P(Y_{Q_i} \mid X_{R_i}, Y_{R_i}, X_{S_i} = 1)] && (2.4)\\
&= \sum_{X_{R_i}, Y_{R_i}} P(Y_{R_i} \mid X_{S_i} = 1) P(X_{R_i} \mid Y_{R_i}) P(Y_{Q_i} \mid X_{R_i}) && (2.5)\\
&= \sum_{X_{R_i} \in (1,-1)} P(Y_{Q_i} \mid X_{R_i}) \int_{Y_{R_i}} P(Y_{R_i} \mid X_{S_i} = 1) P(X_{R_i} \mid Y_{R_i})\, dy && (2.6)
\end{aligned}$$

Equation (5) is due to Markov Chain. Now let's write this equation for each value of $X_{R_i} \in (+1, -1)$:

$$= P(Y_{Q_i} \mid (X_{R_i} = -1)) \int_{Y_{R_i}} P(Y_{R_i} \mid X_{S_i} = 1) P((X_{R_i} = -1) \mid Y_{R_i})\, dy$$
$$+ P(Y_{Q_i} \mid (X_{R_i} = 1)) \int_{Y_{R_i}} P(Y_{R_i} \mid X_{S_i} = 1) P((X_{R_i} = 1) \mid Y_{R_i})\, dy \quad (2.7)$$
$$= P(Y_{Q_i} \mid (X_{R_i} = -1)) \left\{ \int_{Y_{R_i} \geq 0} P(Y_{R_i} \mid X_{S_i} = 1) \cdot 0\, dy + \int_{Y_{R_i} < 0} P(Y_{R_i} \mid X_{S_i} = 1) \cdot 1\, dy \right\}$$
$$+ P(Y_{Q_i} \mid (X_{R_i} = 1)) \left\{ \int_{Y_{R_i} \geq 0} P(Y_{R_i} \mid X_{S_i} = 1) \cdot 1\, dy + \int_{Y_{R_i} < 0} P(Y_{R_i} \mid X_{S_i} = 1) \cdot 0\, dy \right\}$$
$$= P(Y_{Q_i} \mid (X_{R_i} = -1)) \int_{Y_{R_i} < 0} P(Y_{R_i} \mid X_{S_i} = 1)\, dy$$
$$+ P(Y_{Q_i} \mid (X_{R_i} = 1)) \int_{Y_{R_i} \geq 0} P(Y_{R_i} \mid X_{S_i} = 1)\, dy \quad (2.8)$$

Remember the following channel equation $Y_{R_i} = h_{i,1} \cdot X_{S_i} + \eta_{sr_i}$. So, basically the term:

$$\int_{Y_{R_i} < 0} P(Y_{R_i} \mid X_{S_i} = 1)\, dy$$

is the probability of error when the transmitted signal is $X_{S_i} = 1$ (due to the binary signaling). Similarly, the term :

$$\int_{Y_{R_i} \geq 0} P(Y_{R_i} \mid X_{S_i} = 1)\, dy$$

is the probability of success.

Then we can write the Equation 2.8 as follows ($Q(.)$ is the *Q-function*):

$$P(Y_{Q_i} \mid X_{S_i} = 1) = P(Y_{Q_i} \mid (X_{R_i} = -1)) \cdot Q(\frac{h_{i,1}}{\sigma}) + P(Y_{Q_i} \mid (X_{R_i} = 1)) \cdot (1 - Q(\frac{h_{i,1}}{\sigma})) \quad (2.9)$$

Similarly, we can write the denominator in equation 2.1 as:

$$P(Y_{Q_i} \mid X_{S_i} = -1) = P(Y_{Q_i} \mid (X_{R_i} = -1)) \cdot (1 - Q(\frac{h_{i,1}}{\sigma})) + P(Y_{Q_i} \mid (X_{R_i} = 1)) \cdot (Q(\frac{h_{i,1}}{\sigma})) \quad (2.10)$$

Now, let's return back to the equation 2.1:

$$
\begin{aligned}
LLR_{X_{S_i}} &= \ln\left(\frac{P(Y_{Q_i} \mid X_{S_i} = +1)}{P(Y_{Q_i} \mid X_{S_i} = -1)}\right) && (2.11)\\
&= \ln\left(\frac{P(Y_{Q_i} \mid X_{R_i} = -1)\cdot Q(\frac{h_{i,1}}{\sigma}) + P(Y_{Q_i} \mid (X_{R_i} = 1))\cdot(1 - Q(\frac{h_{i,1}}{\sigma}))}{P(Y_{Q_i} \mid X_{R_i} = -1)\cdot(1 - Q(\frac{h_{i,1}}{\sigma})) + P(Y_{Q_i} \mid X_{R_i} = 1)\cdot\underbrace{(Q(\frac{h_{i,1}}{\sigma}))}_{\alpha}}\right)\\
&= \ln\left(\frac{P(Y_{Q_i} \mid X_{R_i} = -1)\cdot\alpha + P(Y_{Q_i} \mid (X_{R_i} = 1))\cdot(1-\alpha)}{P(Y_{Q_i} \mid X_{R_i} = -1)\cdot(1-\alpha) + P(Y_{Q_i} \mid X_{R_i} = 1)\cdot\alpha)}\right) && (2.12)\\
& && (2.13)
\end{aligned}
$$

In other words, α is the **quantization error probability** at relay i.

Now, let us re-write the $LLR_{X_{S_i}}$ using the fact that we have AWGN channels (also remember that we assume no mapping at the relays e.g. $Y_{D_i} = Y_{Q_i}$):

$$
\begin{aligned}
LLR_{X_{S_i}} = \ln&\left(\alpha e^{-\frac{(Y_{D_i}-(-h_{i;2}))^2}{2\sigma^2}} + (1-\alpha)e^{-\frac{(Y_{D_i}-(h_{i;2}))^2}{2\sigma^2}}\right)\\
&- \ln\left((1-\alpha)e^{-\frac{(Y_{D_i}-(-h_{i;2}))^2}{2\sigma^2}} + \alpha e^{-\frac{(Y_{D_i}-(h_{i;2}))^2}{2\sigma^2}}\right) \qquad (2.14)
\end{aligned}
$$

Since $0 < \alpha \leq 1$ and $0 < (1-\alpha) \leq 1$, we can express α as:

$$
\begin{aligned}
\alpha &= e^z \qquad (2.15)\\
(1-\alpha) &= e^t
\end{aligned}
$$

$$
\begin{aligned}
LLR_{X_{S_i}} =\ln&\left(e^z e^{-\frac{(Y_{D_i}-(-h_{i;2}))^2}{2\sigma^2}} + e^t e^{-\frac{(Y_{D_i}-(h_{i;2}))^2}{2\sigma^2}}\right)\\
&- \ln\left(e^t e^{-\frac{(Y_{D_i}-(-h_{i;2}))^2}{2\sigma^2}} + e^z e^{-\frac{(Y_{D_i}-(h_{i;2}))^2}{2\sigma^2}}\right) \qquad (2.16)
\end{aligned}
$$

Now using max-log approximations :

$$\begin{aligned}
LLR_{X_{S_i}} &\sim max\left\{-(\frac{(Y_{D_i}+h_{i;2})^2}{2\sigma^2})+z, -(\frac{(Y_{D_i}-h_{i;2})^2}{2\sigma^2})+t\right\}\\
&\quad - max\left\{-(\frac{(Y_{D_i}+h_{i;2})^2}{2\sigma^2})+t, -(\frac{(Y_{D_i}-h_{i;2})^2}{2\sigma^2})+z\right\}\\
&= max\left\{-(\frac{2Y_{D_i}h_{i;2}}{2\sigma^2})+z, \frac{2Y_{D_i}h_{i;2}}{2\sigma^2}+t\right\} - max\left\{-(\frac{2Y_{D_i}h_{i;2}}{2\sigma^2})+t, \frac{2Y_{D_i}h_{i;2}}{2\sigma^2}+z\right\}\\
&= max\left\{\frac{-Y_{D_i}h_{i;2}}{\sigma^2}+z, \frac{Y_{D_i}h_{i;2}}{\sigma^2}+t\right\} - max\left\{\frac{-Y_{D_i}h_{i;2}}{\sigma^2}+t, \frac{Y_{D_i}h_{i;2}}{\sigma^2}+z\right\}\\
&= \frac{1}{\sigma^2}\overbrace{max\left\{(-Y_{D_i}h_{i;2}+\sigma^2\ln(Q(h_{i;1}/\sigma))), (Y_{D_i}h_{i;2}+(\sigma^2\ln(1-Q(h_{i;1}/\sigma)))\right\}}^{Term1}\\
&\quad - \frac{1}{\sigma^2}\underbrace{max\left\{(-Y_{D_i}h_{i;2}+\sigma^2\ln(1-Q(h_{i;1}/\sigma))), (Y_{D_i}h_{i;2}+(\sigma^2\ln(Q(h_{i;1}/\sigma)))\right\}}_{Term2}
\end{aligned}$$

Now, let's simplify the equation by substituting :

$$\begin{aligned}
b &= \sigma^2\ln(Q(h_{i;1}/\sigma))\\
c &= \sigma^2\ln(1-Q(h_{i;1}/\sigma))
\end{aligned}$$

$$LLR_{X_{S_i}} \sim \frac{1}{\sigma^2}\overbrace{max\{(-Y_{D_i}h_{i;2}+b), (Y_{D_i}h_{i;2}+c)\}}^{\text{Term 1}} - \frac{1}{\sigma^2}\underbrace{max\{(-Y_{D_i}h_{i;2}+c), (Y_{D_i}h_{i;2}+b)\}}_{\text{Term 2}}$$

This is defined on the following two mutually exclusive regions :

Region 1 - $Y_{D_i}h_{i;2} > 0$ **:**

- Term 1: (since $\forall(h_{i;1}/\sigma) \geq 0 \Rightarrow 0 > c \geq b$)

$$max\left\{(-Y_{D_i}h_{i;2}+b), (Y_{D_i}h_{i;2}+c)\right\} = Y_{D_i}h_{i;2}+c$$

- Term 2:

$$max\{(-Y_{D_i}h_{i;2}+c), (Y_{D_i}h_{i;2}+b)\} = \begin{cases} Y_{D_i}h_{i;2}+b & \text{, if } 2Y_{D_i}h_{i;2} > c-b \\ -Y_{D_i}h_{i;2}+c & \text{, if } 2Y_{D_i}h_{i;2} < c-b \end{cases}$$

Then,

$$LLR_{X_{S_i}} \sim \begin{cases} \frac{1}{\sigma^2}(Y_{D_i}h_{i;2}+c-Y_{D_i}h_{i;2}-b) & \text{, if } 2Y_{D_i}h_{i;2} > c-b \\ \frac{1}{\sigma^2}(Y_{D_i}h_{i;2}+c+Y_{D_i}h_{i;2}-c) & \text{, if } 2Y_{D_i}h_{i;2} < c-b \end{cases}$$

Equally:

$$\begin{aligned} LLR_{X_{S_i}} &\sim \begin{cases} \frac{1}{\sigma^2}(c-b) & \text{, if } 2Y_{D_i}h_{i;2} > c-b \\ \frac{2}{\sigma^2}Y_{D_i}h_{i;2} & \text{, if } 2Y_{D_i}h_{i;2} < c-b \end{cases} \\ &= \frac{1}{\sigma^2}min(2Y_{D_i}h_{i;2}, c-b) \end{aligned} \tag{2.17}$$

Region 2 - $Y_{D_i}h_{i;2} < 0$ **:**

- Term 2: (since $\forall(h_{i;1}/\sigma) \geq 0 \Rightarrow 0 > c \geq b$)

$$max\left\{(-Y_{D_i}h_{i;2}+c), (Y_{D_i}h_{i;2}+b)\right\} = -Y_{D_i}h_{i;2}+c$$

- Term 1:

$$max\{(-Y_{D_i}h_{i;2}+b), (Y_{D_i}h_{i;2}+c)\} = \begin{cases} Y_{D_i}h_{i;2}+c & \text{, if } 2Y_{D_i}h_{i;2} > b-c \\ -Y_{D_i}h_{i;2}+b & \text{, if } 2Y_{D_i}h_{i;2} < b-c \end{cases}$$

Then,

$$LLR_{X_{S_i}} \sim \begin{cases} \frac{1}{\sigma^2}(Y_{D_i}h_{i;2}+c-Y_{D_i}h_{i;2}-c) & \text{, if } 2Y_{D_i}h_{i;2} > b-c \\ \frac{1}{\sigma^2}(-Y_{D_i}h_{i;2}+b+Y_{D_i}h_{i;2}-c) & \text{, if } 2Y_{D_i}h_{i;2} < b-c \end{cases}$$

Equally:

$$\begin{aligned} LLR_{X_{S_i}} &\sim \begin{cases} \frac{1}{\sigma^2}(2Y_{D_i}h_{i;2}) & \text{, if } 2Y_{D_i}h_{i;2} > b-c \\ \frac{1}{\sigma^2}(b-c) & \text{, if } 2Y_{D_i}h_{i;2} < b-c \end{cases} \\ &= \frac{1}{\sigma^2}max(2Y_{D_i}h_{i;2}, b-c) \\ &= \frac{1}{\sigma^2}(-min(|2Y_{D_i}h_{i;2}|, |c-b|)), \text{ (due to b-c < 0)} \end{aligned} \tag{2.18}$$

For both regions of interest (Equations 2.18 and 2.17), we can decide the $LLR_{X_{S_i}}$ for the destination decoder as follows:

$$\begin{aligned} LLR_{X_{S_i}} =& \frac{1}{\sigma^2}\text{sign}(Y_{D_i}h_{i;2}) \min\left(|2Y_{D_i}h_{i;2}| \text{ , } |b-c|\right) \\ =& \frac{1}{\sigma^2}\text{sign}(Y_{D_i}h_{i;2}) \min\left(|2Y_{D_i}h_{i;2}| \text{ , } \sigma^2|\ln(\frac{1-Q(h_{i;1}/\sigma)}{Q(h_{i;1}/\sigma)})|\right) \\ =& \frac{1}{\sigma^2}\text{sign}(Y_{D_i}h_{i;2}) \min\left(|2Y_{D_i}h_{i;2}| \text{ , } \sigma^2|\ln(\frac{1}{Q(h_{i;1}/\sigma)}-1)|\right) \end{aligned}$$

Now, for the sake of implementation simplicity, let's approximate the term $\sigma^2|\ln\left(\frac{1}{Q(h_{i;1}/\sigma)}-1\right)|$ using Chernoff bound:

$$0 \leq Q(\frac{h_{i;1}}{\sigma}) \leq \frac{1}{2}e^{-\left(\frac{h_{i;1}}{2\sigma}\right)^2}$$

So;

$$\begin{aligned}
\sigma^2|\ln(\frac{1}{Q(h_{i;1}/\sigma)} - 1)| &\geq \sigma^2|\ln(2e^{\left(\frac{h_{i;1}}{2\sigma}\right)^2} - 1) \\
&\geq \sigma^2|\ln(2e^{\left(\frac{h_{i;1}}{2\sigma}\right)^2}) \\
&= \sigma^2|\ln(e^{0.69314\ldots}e^{\left(\frac{h_{i;1}}{2\sigma}\right)^2}) \\
&\geq \sigma^2(\frac{|h_{i,1}|^2}{4\sigma^2} + 0.69314) \\
&\geq \frac{|h_{i,1}|^2}{4} + 0.69314 \cdot \sigma^2
\end{aligned}$$

2.3 Conclusion

Equation 2.19 can be interpreted as follows:

$$LLR_{X_{S_i}} = \frac{1}{\sigma^2}\mathrm{sign}(Y_{D_i}h_{i;2})\ \min(|2Y_{D_i}h_{i;2}|, \frac{|h_{i,1}|^2}{4}) \tag{2.19}$$

- Take the received and compensated signal ($|2Y_{D_i}h_{i;2}|$) at the destination,
- Find the magnitude of the first channel ($|h_{i,1}|^2/4$) and compare these two.
- Use the minimum of these as the magnitude of the LLR,
- and the sign of it always comes from the ($|2Y_{D_i}h_{i;2}|$) (second link, Relay-Destination).